Complex Analysis: Final Exam

Aletta Jacobshal 02, Wednesday 1 February 2017, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of your exam sheet and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (15 points)

Evaluate

$$\text{pv}\int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+1)}dx$$

using the calculus of residues.

Solution

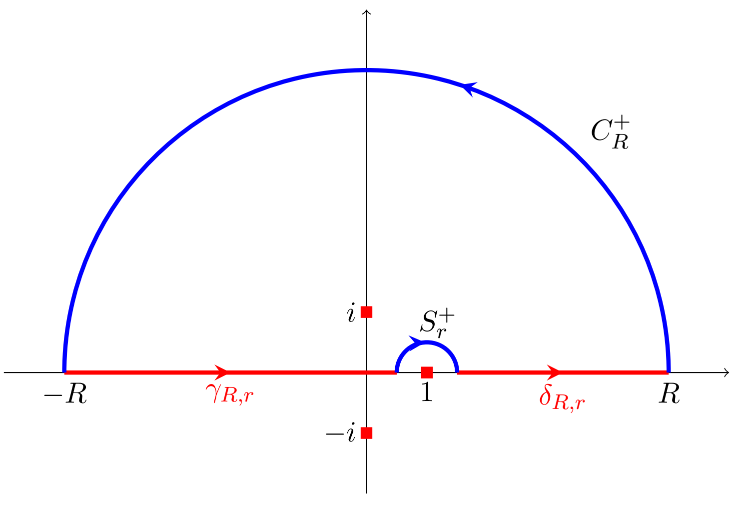
By definition,

$$\begin{aligned} I &= \text{pv}\int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+1)}dx \\ &= \lim_{\substack{R\to\infty \\ r\to 0^+}} \left(\int_{-R}^{1-r} \frac{1}{(x-1)(x^2+1)}dx + \int_{1+r}^{R} \frac{1}{(x-1)(x^2+1)}dx \right) \\ &= \lim_{\substack{R\to\infty \\ r\to 0^+}} I_{R,r}. \end{aligned}$$

To compute this integral we consider the closed contour

$$C_{R,r} = \gamma_{R,r} + S_r^+ + \delta_{R,r} + C_R^+,$$

shown below.



We have

$$\begin{aligned} I_{R,r} &= \int_{-R}^{1-r} \frac{1}{(x-1)(x^2+1)}dx + \int_{1+r}^{R} \frac{1}{(x-1)(x^2+1)}dx \\ &= \left(\int_{\gamma_{R,r}} + \int_{\delta_{R,r}}\right) f(z)dz, \end{aligned}$$

where

$$f(z) = \frac{1}{(z-1)(z^2+1)}.$$

Therefore,

$$\int_{C_{R,r}} f(z)dz = I_{R,r} + \int_{S_r^+} f(z)dz + \int_{C_R^+} f(z)dz.$$

For $R > 2$ and $r < 1$ we have

$$\int_{C_{R,r}} f(z)dz = 2\pi i \operatorname{Res}(i) = -\frac{\pi}{2}(1+i),$$

where we used that

$$\operatorname{Res}(i) = \lim_{z\to i}(z-i)\frac{1}{(z-1)(z-i)(z+i)} = \lim_{z\to i}\frac{1}{(z-1)(z+i)} = \frac{1}{2i(i-1)} = -\frac{1}{4i}(1+i).$$

At the limit $r \to 0^+$ we have

$$\lim_{r\to 0^+}\int_{S_r^+} f(z)dz = -\pi i \operatorname{Res}(1) = -\frac{\pi}{2}i,$$

where we used that

$$\operatorname{Res}(1) = \lim_{z\to 1}(z-1)\frac{1}{(z-1)(z^2+1)} = \lim_{z\to 1}\frac{1}{z^2+1} = \frac{1}{2}.$$

Moreover, since the degree of the denominator is 3 and the degree of the numerator is 0 we have

$$\lim_{R\to\infty}\int_{C_r^+} f(z)dz = 0.$$

Then taking the limits $R \to \infty$ and $r \to 0^+$ we get

$$-\frac{\pi}{2}(1+i) = I - \frac{\pi}{2}i + 0,$$

giving

$$I = -\frac{\pi}{2}.$$

Question 2 (15 points)

Show that if f is analytic in the closed disk $|z| \leq 2$ and if $|f(z)| < 1$ on the circle $|z| = 1$, then the equation $f(z) = z^n$ has exactly n solutions (counting multiplicity) in the open disk $|z| < 1$.
Hint: Rouché's theorem; the conditions for applying the theorem must be explicitly stated and verified.

Solution

The functions $f(z)$ and z^n are both analytic on and inside the unit circle.

The number of zeros of z^n inside the unit circle, counting multiplicity, is $N_0(z^n) = n$.

Moreover, on the unit circle we have

$$|-f(z)| < 1 = |z^n|.$$

Therefore, applying Rouché's theorem we get

$$N_0(z^n) = N_0(z^n - f(z)) = n,$$

which implies that the equation $z^n = f(z)$ has exactly n solutions (counting multiplicity) inside the unit circle.

Question 3 (15 points)

Represent the function

$$f(z) = \frac{z+1}{z-1},$$

(a) (8 points) as a Taylor series around 0 and find its radius of convergence;

Solution

$$\begin{aligned}\frac{z+1}{z-1} &= -(z+1)(1+z+z^2+z^3+z^4+\cdots) \\ &= -1-2z-2z^2-2z^3-2z^4-2z^5-\cdots,\end{aligned}$$

where we used the geometric series which converges for $|z| < 1$. The only singularity of $(z+1)/(z-1)$ is at $z = 1$ which is at a distance $|z| = 1$ from 0. Therefore, the radius of convergence is 1.

(b) (7 points) as a Laurent series in the domain $|z| > 1$.

Solution

Since $|z| > 1$, that is $|1/z| < 1$, we have

$$\frac{z+1}{z-1} = \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right)\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right).$$

Therefore,

$$\frac{z+1}{z-1} = 1+\frac{2}{z}+\frac{2}{z^2}+\frac{2}{z^3}+\frac{2}{z^4}+\frac{2}{z^5}+\cdots$$

Question 4 (15 points)

At which points is the function

$$f(z) = x^2+y^2+2ixy,$$

differentiable? Compute the derivative of $f(z)$ at these points.

Solution

We check the partial derivatives, where we write $u(x, y) = x^2 + y^2$ and $v(x, y) = 2xy$. We have

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = 2y,$$
$$\frac{\partial v}{\partial x} = 2y, \qquad \frac{\partial v}{\partial y} = 2x.$$

All partial derivatives exist and are continuous for all $x + iy \in \mathbb{C}$. Then the Cauchy-Riemann equations give $2x = 2x$ and $2y = -2y$, that is, $y = 0$ and $x \in \mathbb{R}$.

The derivative is given by

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2x - 2iy = 2x.$$

Question 5 (15 points)

Consider the function

$$f(z) = \frac{\sin z}{z^2}.$$

(a) (6 points) Determine the singularities of $f(z)$ and their type (removable, pole, essential; if pole, specify the order).

Solution

The function has a singularity at $z = 0$. The Taylor series for $\sin z$ is

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots,$$

therefore,

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{1}{3!}z + \frac{1}{5!}z^3 - \cdots.$$

Therefore, $z = 0$ is a pole of order 1.

Alternatively, we notice that

$$\lim_{z \to 0} z\frac{\sin z}{z^2} = 1,$$

implying that $z = 0$ is a pole of order 1.

(b) (9 points) Show that $f(z)$ does not have an antiderivative in $\mathbb{C} \setminus \{0\}$. *Hint: Compute the integral of f along the unit circle.*

Solution

If a function has an antiderivative in a domain D then its integral over any closed contour in D must be 0. We compute

$$\int_C \frac{\sin z}{z^2} dz,$$

where C is the positively oriented unit circle. From the generalized Cauchy formula we have

$$\int_C \frac{\sin z}{z^2} dz = 2\pi i (\sin z)'|_{z=0} = 2\pi i \cos 0 = 2\pi i.$$

Question 6 (15 points)

(a) (8 points) Prove that

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

where $z = x + iy$.

Solution

We compute

$$\begin{aligned} \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\ &= \frac{1}{2}(e^{ix-y} + e^{-ix+y}). \end{aligned}$$

Moreover,

$$\begin{aligned} \cos x \cosh y - i \sin x \sinh y &= \frac{1}{2}(e^{ix} + e^{-ix})\frac{1}{2}(e^y + e^{-y}) - i\frac{1}{2i}(e^{ix} - e^{-ix})\frac{1}{2}(e^y - e^{-y}) \\ &= \frac{1}{4}(e^{ix+y} + e^{ix-y} + e^{-ix+y} + e^{-ix-y}) \\ &\quad - \frac{1}{4}(e^{ix+y} - e^{ix-y} - e^{-ix+y} + e^{-ix-y}) \\ &= \frac{1}{2}(e^{ix-y} + e^{-ix+y}). \end{aligned}$$

This proves the required relation.

(b) (7 points) Prove that the function

$$u(x, y) = \cos x \cosh y,$$

is harmonic in $\mathbb{R}^2$ and find a harmonic conjugate of $u(x, y)$.

Solution

The given function $u(x, y)$ is harmonic because it is the real part of the analytic function $\cos z$.

A harmonic conjugate for $u(x, y)$ is then the imaginary part of $\cos z$, that is,

$$v(x, y) = -\sin x \sinh y.$$